Criterium for the index theorem on the lattice

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Abstract. We study how far the Index Theorem can be extrapolated from the continuum to finite lattices with finite topological charge densities. To examine how the Wilson action approximates the Index theorem, we specialize in the lattice version of the Schwinger model. We propose a new criterion for solutions of the Ginsparg-Wilson Relation constructed with the Wilson action. We conclude that the Neuberger action is the simplest one that maximally complies with the Index Theorem, and that its best parameter in $d = 2$ is $m_0 = 1.1 \pm 0.1$.

1. INTRODUCTION

One of the constraints that lead to the Nielsen-Ninomiya [1] no-go theorem, is the chiral invariance of the Dirac action D for massless fermions on the lattice. Under certain conditions [1] , the spectrum of fermions ψ would suffer from doubling, and the axial anomaly would cancel. This has been, from the onset, a recurrent problem of Lattice QCD [2]. In order to recover the axial anomaly, at least at the pertubative level, Ginsparg and Wilson [3] derived a relation, the Ginsparg-Wilson Relation (GWR) which explicitly breaks the standard chiral symmetry,

$$D\gamma_5 + \gamma_5 D = D\gamma_5 RD \, , \tag{1}$$

where R is a matrix proportional to the lattice spacing a (in most of this talk we will consider the case where R is a simple number). Recently Lüsher [4] proved that chiral invariance of an action D which complies with the GWR can be recovered in an extended form,

$$\delta\psi = \gamma_5(I - \frac{1}{2}RD)\psi \, , \quad \delta\bar{\psi} = \bar{\psi}(I - \frac{1}{2}DR)\gamma_5 \, . \tag{2}$$

Thus a conjecture appeared in the literature suggesting that it may be possible to overcome the Nielsen-Ninomiya no-go theorem on the lattice and to fully simulate, without doubling, the chiral symmetry of QCD (see [5] for a recent review). This includes PCAC [6] and the axial anomaly [7]. In particular the Atiyah-Singer [8] Index Theorem has a Hasenfratz-Laliena-Niedermayer [9] version on the lattice,

$$n_- - n_+ = q \, , \tag{3}$$

where $n_{-(+)}$ is the number of zero modes of the Dirac action D with negative (positive) chirality. q is an *integer* topological charge of the gauge field configuration, defined with

the charge density, a function of the lattice site $\vec{n}$,

$$\rho(\vec{n}) = \frac{1}{2} tr \bigg|_{colour,Dirac} \{\gamma_5 R D(\vec{n},\vec{n})\} \ . \tag{4}$$

Therefore each GWR Dirac action *defines* the density (4) and the index (3). Moreover examples of lattice actions have been found where this index is nontrivial and coincides with a charge independently defined from a topological gauge density. For instance the fixed point topological density [10] coincides, on all gauge configurations, with the density (4) defined with the fixed point Dirac operator. The recent overlap GWR solution of Neuberger [11] also solves the GWR and complies with the Index Theorem. The Neuberger solution was checked in dimension $d = 2$ for smooth gauge configurations [12, 13] , and in dimension $d = 4$ in the continuum limit [14, 15]. Thus the GWR, together with these recent results, essentially solve the problem of the topological Index Theorem on the lattice.

Nevertheless we remark that the definition of the Dirac operator and the definition of the topological charge on the lattice are not unique. Moreover the topological charge on the lattice is not conserved [16]. This suggests that different GWR Dirac operators and topological densities produce different indexes when we depart from the continuum limit, to a finite lattice with finite charge density. In this talk we address the *extremum* problem of finding the lattice Dirac operator which provides the best index (3). Because this is a difficult problem, we specialize in lattice actions which are constructed with the Wilson action [2, 17]. In order to compare the index of different Dirac operators, we first choose a single topological charge density. Although this is an arbitrary definition, let us choose the simplest near neighbor lattice topological charge density which discretizes the continuum density $\varepsilon_{\mu\nu\dots}F^{\mu\nu}\cdots$ and which sums to an integer total topological charge. Then we search for the GWR Dirac operator which index (3) is closer to this simplest topological charge.

The arbitrariness in the choice of our lattice operators is partly constrained by locality. Locality is a crucial issue to connect the lattice to the Quantum Field Theories in the continuum. Only local actions are universal. Moreover it has been conjectured [5, 18] that local GWR solutions have better topological properties than non-local ones. Only topological configurations which are larger than the range of the couplings in D would produce a number of zero modes n_- different from n_+ in agreement with eq. (3). This conjecture is certainly correct in the case of the Zenkin [19, 20] action. The Zenkin action is non-local [21], and although it complies with eq. (1), it does not comply with eq. (3). Thus we will restrain from using non-local definitions for the lattice action and for the lattice charge density.

In this talk we investigate in detail the eigenvalues of some lattice actions, with the aim to clarify how the Index Theorem is extrapolated from the continuum to finite lattices with finite topological charge densities. We also propose a new criterion for GWR solutions constructed with the Wilson action. In Section 5 we study in detail the effects of topological gauge configurations on the eigenvalues of the Wilson action for the discrete version of the Schwinger model. In Section 2 we review properties of lattice actions. In Section 3 we address the topological Abelian charge in $d = 2$. In Section 4

we review the Wilson action. In Section 6 we present our criterion. Finally in Section 7 we conclude.

2. SOME PROPERTIES OF LATTICE ACTIONS

A plausible property of lattice actions is γ_5 hermiticity. For instance the Wilson action is γ_5 Hermitean. When an action verifies $D^\dagger = \gamma_5 D \gamma_5$, we can show that D and $D^\dagger$ have the same eigenvalues λ,

$$
\begin{aligned}
D\,v &= \lambda\,v \\
\Leftrightarrow\ D^\dagger\,\gamma_5\,v &= \lambda\,\gamma_5\,v\,,
\end{aligned}
\tag{5}
$$

and that the eigenvalues of D and $D^\dagger$ are mapped by γ_5. When we conjugate eq (5), we find that for every right eigenvalue there is a left eigenvalue λ^* with eigenvector $v^\dagger\,\gamma_5$, and so the conjugate transformation $\lambda \to \lambda^*$ leaves the spectrum invariant. The spectrum of eigenvalues λ is symmetric with respect to the real axis of the Argand plot.

We proceed to define chirality in the lattice. Eq. (5) can be used to show that the γ_5 is involved in a extended orthogonality condition,

$$
\left(\lambda_2^* - \lambda_1\right) v_2^\dagger\,\gamma_5\,v_1 = 0\,.
\tag{6}
$$

A possible definition for the chirality of an eigenvector v is,

$$
\chi = \frac{v^\dagger\,\gamma_5\,v}{v^\dagger\,v},
\tag{7}
$$

and eq. (6) shows that the complex eigenvalues have a vanishing chirality, $\chi = 0$. Only the real eigenvalues may have a non vanishing chirality.

We now study properties of GWR solutions. The GWR implies that if v is a zero mode of the GWR action $Dv = 0$, then $D\gamma_5 v = 0$. Thus we can use $1 - \gamma_5$ and $1 + \gamma_5$ to decompose the Kernel in a set of left vectors and a set of right vectors. This shows that the zero modes have a chirality $\chi = \pm 1$. Moreover we can verify that if D is a GWR action, then $-D + 2/R$ also complies with the GWR. Thus the eigenvalues $2/R$ of D also have a chirality $\chi = \pm 1$.

It is also convenient to define an intermediate matrix,

$$
V = RD - I \Leftrightarrow D = \frac{1}{R}(I + V)
\tag{8}
$$

because the GWR is equivalent to $V^{-1} = \gamma_5 V \gamma_5$. In the case that V is γ_5 Hermitean, we also obtain that $V^{-1} = V^\dagger$, and this shows that the eigenvectors of V belong to the unitary circle of the complex Argand plane. The eigenvalues of D are in a circle with center $1/R + 0i$ and radius $1/R$. We will be interested in the -1 eigenvalues of V which correspond to zero modes of D.

Moreover the crucial property for the Index Theorem has been show,

$$
\frac{R}{2} tr\{\gamma_5 D\} = n_- - n_+\,,
\tag{9}
$$

both with analytical [4] and algebraic [14] methods.

3. TOPOLOGICAL CONFIGURATIONS IN D=2

The $d = 2$ case is very convenient for the simple study of discrete topology. The Euclidean Dirac matrices $\gamma_{1\to 5}$ of $d = 4$ are now replaced by the Pauli matrices $\sigma_{1\to 3}$. The $d = 2$ charge density is similar to the rotational of the vector potential,

$$\rho(x) = F_{12}(x) \,, \tag{10}$$

it is similar to a magnetic field. On the lattice this magnetic field density is extracted from the plaquette,

$$P(\vec{n}) = U(\vec{n})_1 U(\vec{n} + \hat{1})_2 U(\vec{n} + \hat{2})_1^* U(\vec{n})_2^* \tag{11}$$

and a possible definition of topological charge density is,

$$\rho_1(\vec{n}) = \frac{1}{2\pi} \arg P(\vec{n}) \,, \tag{12}$$

where $-1/2 < \rho_1(\vec{n}) < 1/2$. This definition of the topological density is not unique, the necessary condition is that it must reproduce the correct continuum limit. The definition (12) of the topological charge is particularly interesting because it is quantized. It is clear that the total topological charge of the lattice, which is defined with,

$$q = \sum_{\vec{n}} \rho(\vec{n}) \,, \tag{13}$$

is an integer. The charge q_1 is the sum of the decimal part of a set of numbers, and these numbers have a vanishing sum, therefore q_1 is an integer. This is similar to $d = 2$ compact QED, where the magnetic monopoles arise naturally on the lattice. Here the topological charge is equivalent to a magnetic flux through the toroidal $d = 2$ lattice.

We now construct non-trivial gauge configurations. In this work we are particularly interested in extrapolating continuum properties to finite lattices. We specialize to gauge configurations with uniform topological charge density. These configurations are unique, except for gauge transformations. In the $N \times N$ lattice a simple configuration is built with the prescription [12] of Chiu,

$$
\begin{aligned}
U_j(\vec{n}) &= \exp\left[iA_j(\vec{n}) \right] \,, \\
A_1(n_1, n_2) &= -2\pi Q \frac{n_2 - 1}{N^2} \,, \\
A_2(n_1, N) &= 2\pi Q \frac{n_1 - 1}{N} \,,
\end{aligned}
\tag{14}
$$

where the undefined $A_j(n)$ are zero. With this definition (14) and in the particular case of integer Q, the topological density is constant. If $-N^2/2 < Q < N^2/2$ we find that $\rho_1 = Q/N^2$ and that the topological charge is $q_1 = Q$.

The case where Q is an arbitrary continuous parameter can be used when we are interested in an interpolation between the cases of uniform density. Then the density is uniform, except at the single point (N,N). The gauge configuration (14) is periodic in Q, with a period of N^2. In the relevant range $-N^2/2 < Q < N^2/2$ we find that the topological charge q_1 of eq. (12) is a step-like function of Q,

$$q_1 = \text{int}\left(Q\frac{N^2-1}{N^2} + \frac{1}{2}\right) .$$ (15)

4. THE WILSON ACTION

The Wilson [2] action is the simplest and most widely used action in lattice field theory. It also provides a good example to observe roots and doubles,

$$
\begin{aligned}
{}^{w}D &= \frac{1}{a}\sum_j \gamma_j C_j + \frac{2r}{a}\sum_j B_j , \\
C_{j\vec{n},\vec{n}'} &= \frac{U(\vec{n})_j \delta_{n+\hat{j},n'} - U(\vec{n}')_j^\dagger \delta_{\vec{n},\vec{n}'+\hat{j}}}{2} , \\
B_{j\vec{n},\vec{n}'} &= \frac{2\delta_{\vec{n},\vec{n}'} - U(\vec{n})_j \delta_{n+\hat{j},\vec{n}'} - U(\vec{n}')_j^\dagger \delta_{\vec{n},\vec{n}'+\hat{j}}}{4} ,
\end{aligned}
$$ (16)

where r is a parameter of the order of 1. The vector γ_j term of eq. (16) is the naive massless Dirac action on the lattice, where the derivative C_j is computed by means of finite differences. In the limit of free fermions, which corresponds to $U_j = 1$, all these operators commute. In this case the Fourier transform of the above matrices depend on a single momentum,

$$
\begin{aligned}
I(k) &= 1 , \\
C_j(k) &= i\sin k_j , \\
B_j(k) &= \sin^2\frac{k_j}{2} .
\end{aligned}
$$ (17)

In the free case, and in d dimensions, $C_j(k)$ has 2^d roots at all possible combinations of $k_j = 0$ and $k_j = \pi$, see eq. (17). This is the source of the doubler problem because only the root at vanishing k is physical. Wilson [2] included the scalar B_j term in eq. (16) to remove the unwanted doubles.

However this scalar term B_j is not chiral invariant. Chiral invariance can be implemented with the Lüscher transformation. However Horvath showed that the RD operator can at most [22] be exponentially local, therefore the ${}^{w}R$ of the GWR is not a simple number in this case. From the GWR relation, see eq. (1) we get,

$$ {}^{w}D\,{}^{w}R = I + {}^{w}D\frac{1}{{}^{w}D^\dagger}, $$ (18)

where the Wilson action $^{w}D^{\dagger}$ is in general invertible, see Section 5. We can remark that the operator in eq. (18) is a solution of the GWR, however it is not exponentially local. In this case the nontrivial operator of the Lüscher transformation,

$$\gamma_5 \left(I - \frac{1}{2} \, {}^{w}R^{w}D \right) = \frac{1}{2} \left(\gamma_5 + {}^{w}D^{-1}\gamma_5^{w}D \right), \tag{19}$$

is also non-local. Moreover $^{w}D^{w}R$ is not γ_5 Hermitean. This excludes both the Wilson Action and the $^{w}D^{w}R$ as the best candidate to study exact chiral symmetry on the lattice.

5. EIGENVALUES OF THE WILSON ACTION

Here we depart from the free case and study the eigenvalues of the Wilson action for topological gauge configurations . This continues reference [23]. The Wilson action ^{w}D is the simplest and most used lattice action, see Section 4. Our framework is the massless Wilson action with parameters $a = r = 1$, see eqs (16), in a simple 2-dimensional U(1) gauge theory which is a discrete version of the Schwinger model, see Section 3. In the continuum the Wilson action [24] exactly complies with the Index Theorem (3), but for finite lattices it does not. Our aim in this section is to extrapolate the topological charge from the continuum to the lattices of different size $N \times N$, and to study how the Wilson Action deviates from the Index theorem.

We encounter the problem that the topological charge and the topological density on the lattice are not uniquely defined. This problem is not present in the continuum, where the topological density and its derivatives are well defined. This implies that a lattice action close to the continuum limit is only allowed to have a topological charge density both small and nearly constant. However we are interested, both physically and mathematically, in extrapolating the Index Theorem to the case of finite lattices, with finite topological charge. And then we want to investigate how far the topological density can be increased without spoiling the index theorem. To remain as close as possible to the continuum definitions, we momentarily specialize to very smooth gauge configurations. In particular we study the Wilson action in configurations with constant plaquette density, see Section 3, which is equivalent to a constant topological charge density.

First we study some general properties of the Wilson action. Because it is γ_5 Hermitean, the spectrum of its eigenvalues, here denoted λ, is symmetric with respect to the real axis of the Argand plot, see Section 2. Moreover in $d = 2$ it turns out that in the case of even lattices, the spectrum of $^{w}D - 2I$ in the Argand plot is also symmetric with respect to the imaginary axis. The transformation $(\lambda - 2) \rightarrow -(\lambda - 2)^{*}$ leaves the spectrum invariant (see for instance Figs. 1,2,3). So we will use even $N \times N$ lattices with $N \geq 4$ from now on, for simplicity. We also remark that $0 < \text{real}(\lambda) < 4$ for any non vanishing gauge configuration, thus ^{w}D is invertible (except in the free quark case). In what concerns $\gamma_5(^{w}D - 2I)$ we observe that its eigenvalues are exactly symmetric with respect to the origin. This implies for instance that the traces $\text{tr}\left[\gamma_5 (^{w}D - 2I)^{-1}\right]$ and $\text{tr}\left[\gamma_5 \, ^{w}D\right]$ both vanish for any finite N. This is not the case of the trace $\text{tr}\left[\gamma_5 \, ^{w}D^{-1}\right] / \mathscr{Z}$ which is non vanishing and has been proposed as a topological index [25], where $\mathscr{Z}$ is a constant

normalizing factor. However we will not further study this proposed topological charge because it clearly is neither local nor integer.

We now study quantitatively the real eigenvalues and the topological properties of the Wilson action ^wD. We are interested in eigenvalues $\lambda \simeq 0$ with chirality $\chi \simeq \pm 1$, see Section 2. The non vanishing chirality implies that the relevant eigenvalues are the real eigenvalues. The B_j term was introduced in the free action by Wilson to separate the four roots of the C_j, see Section 4. In the free limit this results in four real eigenvalues: one root at 0 , two degenerate eigenvalues equal to 2 and one eigenvalue equal to 4. Here we find that quite a similar separation exists in the quadruplets of real eigenvalues that occur in gauge configurations with a finite topological charge (see Figs. 1,2,3). Moreover the chirality of these real eigenvalues approximately agree with the Index Theorem. In particular we observe that,

- The number of real eigenvalues is always a multiple of four, $4|q_2|$. The integer $q_2 = -\text{sign}(\chi)|q_2|$ is a possible definition of the topological charge of the gauge configuration, where χ is the chirality of the smallest real eigenvalue.
- In the present case of a constant topological density, $q_2 = q_1$ if $|\rho_1|$ is smaller than half of the maximum possible charge density. The integer topological charge q_1 and its topological density ρ_1 are defined in Section 3.
- The real eigenvalues and the corresponding chiralities are close to integers. The difference is proportional to a small number, $\varepsilon = |q_2|/N^2$. The $4|q_2|$ eigenvalues can be divided in the following 3 sets of eigenvectors, with eigenvalues,

$$
\begin{aligned}
\lambda &= 0 + 3.0\varepsilon + o(\varepsilon^2) &, \quad \chi &= [-1 + o(\varepsilon^2)]sign(q_2), \\
\lambda &= 2 + o(\varepsilon^2) &, \quad \chi &= [1 - 2.0\varepsilon + o(\varepsilon^2)]sign(q_2), \\
\lambda &= 4 - 3.0\varepsilon + o(\varepsilon^2) &, \quad \chi &= [-1 + o(\varepsilon^2)]sign(q_2),
\end{aligned}
\tag{20}
$$

which contain respectively $|q_2|$, $2|q_2|$ and $|q_2|$ eigenvalues.
- When $|q_2|$ approaches the maximum value of the order of $n^2/4$, these 3 sets of eigenvalues spread out to 3 finite intervals, and eventually they overlap. It turns out that right before these intervals overlap, they just leave a gap at $[1.0, 1.2]$.
- In the continuum limit of $N \to \infty$, and for very smooth topological objects, all the different $\varepsilon \to 0$. So we find that in this limit of large N, and from the view point of zero modes, the Index Theorem is verified,

$$
n_- - n_+ = q_2 .
\tag{21}
$$

This result agrees with the proofs of Fujikawa [14] and Adams [15].

If we now leave the case of constant topological density, it occurs that $|\rho_1| < 0.22$ (we mean that the absolute value of the topological density ρ_1 is smaller than 0.22 in any plaquette of the lattice) is a sufficient condition to enforce that $q_2 = q_1$. We verified this crucial result with a large number of randomly generated gauge configurations, for all possible q_1. All the remaining properties, that we just detailed, of the eigenvalues of the Wilson action are also maintained when $|\rho_1| < 0.22$. This suggests that it is possible to extrapolate the continuum limit to a significant and finite subinterval of the range $-1/2 < \rho_1 < 1/2$.

To illustrate how the complex eigenvalues transform in real ones, we arbitrarily interpolate between different constant charge densities with a continuous variation of the topological parameter Q (see Section 3). It turns out that for some particular values of Q, which produce a large $|\rho_1|$ at the point (N,N), two opposite pairs of complex eigenvalues (with 0 chirality) quite suddenly transform into two opposite pairs of real eigenvalues, and then these real eigenvalues continuously increase their chirality $|\chi|$. This transition point is continuous, but at this precise value of Q the velocity of the eigenvalues in the Argand plot is infinite. At this point the topological number q_2 steps up (down). The opposite also happens, at some other particular values of Q, where 4 real eigenvalues suddenly transform into complex eigenvalues. The different topological charges can also be computed (see Section 3). and the observed pattern is quite general. The charge q_2, defined with the real eigenvalues of the Wilson action, and the charge q_1, defined with the plaquette density P in Section 3, are quite close only up to half of the maximum possible topological q_1. This verifies our empirical rule of $|\rho_1| < 0.22$.

In Figs. 1,2,3 we show a sequence of the eigenvalues of the Wilson action for consecutive values of the interpolating topological parameter Q. A simple 4×4 lattice is used and the eigenvalues are displayed in the Argand plan.

6. CRITERION

We aim to understand how the Index Theorem extrapolates from the continuum limit to finite lattices with finite topological charge densities. In d=2, and providing that the topological density $|\rho_1| < 0.22$, we show that the Wilson action ^wD has the correct number of small real eigenvalues to comply with the index theorem. The problem is that these eigenvalues are not exactly vanishing and that their chirality is not exactly ± 1 . The deviations to the index theorem are of order $\varepsilon = |q_1|/N^2$, where q_1 is the integer topological charge. We also prove that the GWR Neuberger solutions either project these approximate zero modes of ^wD on the origin, with correct chirality, or send them close to the remote other end $2/R$ of the unitary circle.

This motivates a new criterion for GWR solutions ^{gwr}D with index identical to q_1 (and γ_5 Hermitean, constructed with the Wilson action), which relies in the function $f(\lambda)$. This function $f(\lambda)$ is straightforwardly defined by replacing ^wD and $^wD^\dagger$ by the same real number λ in the expression for ^{gwr}D. We map $^{gwr}D(^wD,^wD^\dagger) \rightarrow f(\lambda) =^{gwr} D(\lambda,\lambda)$.

The $f(\lambda)$ is a trivial case of a GWR solution. Moreover the γ_5 hermiticity implies that $f(\lambda)$ is real. We thus conclude that $f(\lambda) = 0$ or $f(\lambda) = 2/R$.

In the free case ^wD and $^wD^\dagger$ commute, and have the correct real eigenvalues at 0, while the doubles correspond to the eigenvalues 2 or 4. We remark that the doubles correspond to pathological eigenvectors, with alternating signs in neighboring lattice sites. We conclude that, unless the fermion fields are redefined, a correct GWR solution must produce $f(0) = 0$, $f(2) = 2/R$ and $f(4) = 2/R$.

Studying small topological densities ρ_1 with finite q_1, we can further show that $f([0,3.0\varepsilon]) = 0$, $f(2) = 2/R$, $f([4 - 3.0\varepsilon,4]) = 2/R$. In the case of small topological densities, ^wD and $^wD^\dagger$ have the same eigenvalues, but they do not exactly have the same eigenvectors. Nevertheless the relevant small real eigenvalues λ of the Wilson action,

have a chirality $\chi = [-1 + o(\varepsilon^2)]\mathrm{sign}(q_2)$ and this implies that the difference between the ${}^wD$ eigenvector v and the ${}^wD^\dagger$ eigenvector $\gamma_5 v$ is very small, of order ε^2. This shows that ${}^{gwr}Dv = f(\lambda)v + o(\varepsilon^2)$, the correct eigenvector v' of ${}^{gwr}D$ is quite close to v (and to $\gamma_5 v$). *If ${}^{gwr}D$ complies with the index theorem,* we can assume that this eigenvector v' is a zero mode, because v is already quite close to the correct fermionic zero mode. Therefore $f(\lambda) \simeq 0$. Moreover we can only have $f(\lambda) = 0$ or $2/R$. This implies that $f(\lambda) = 0$. Because $\lambda = 0 + 3.0\varepsilon + o(\varepsilon^2)$ we find that *f projects the interval* $[0, 3.0\varepsilon]$ *on the origin* 0. *Inversely, if* $f(\lambda) = 0$, ${}^{gwr}D$ has $|q_1|$ eigenvectors close to $f(\lambda) = 0$. Moreover they all have similar chirality $\chi = [-1 + o(\varepsilon^2)]\mathrm{sign}(q_2)$. Even if these eigenvectors mix, which is natural because they are very close, their chirality will remain close to $\mathrm{sign}(q_2)$, and finite. *Therefore these* $|q_1|$ *eigenvalues are real, vanishing, with the correct chirality, and in the right number to comply with the index theorem*. Similarly we can show that f projects on $2/R$ both a small neighborhood of the point 2 and the interval $[4 - 3.0\varepsilon, 4]$. If we assume that the complex eigenvalues of the Wilson action are irrelevant to the index because they have vanishing chirality, we conclude that the condition $f([0, 3.0\varepsilon]) = 0$, $f(2) = 2/R$, $f([4 - 3.0\varepsilon, 4]) = 2/R$ is both a necessary and a sufficient condition for any γ_5 Hermitean GWR solution, constructed with the Wilson action, and free from doubling, to comply with the index theorem in the case of small topological density.

In the case of a finite topological charge density the three sectors of real eigenvectors of the Wilson that include 0, 2 and 4 spread out. Analytically it is hard to find how far the exact criterion that we just derived for small ε can be extended to finite ε. Nevertheless our numerical study of the Wilson action shows that up to moderate densities $|\rho_1| < 0.22$, the sectors including 0, 2 and 4 remain separated. For instance the sector of 0, with chiralities $\chi \simeq - \mathrm{sign}(q_1)$, and the sector of 2, with chiralities $\chi \simeq + \mathrm{sign}(q_1)$, are separated by a gap located at $\lambda_0 = 1.1 \pm 0.1$. It is thus quite plausible that a correct GWR solution should project the smaller $|q_2|$ real eigenvalues λ of ${}^wD$ on 0, and that the remaining $3|q_2|$ real eigenvalues should be projected onto the other end of the circle, close to $2/R$. We may now determine which is the function $f(\lambda)$ that maximally complies with the index theorem, up to the highest possible topological density 0.22. This $f(\lambda)$ is a step function that jumps precisely at the gap $\lambda_0 = 1.1 \pm 0.1$.

Finally when the density is large, $|\rho_1| > 0.22$ or more at one or many plaquettes of the lattice, the index theorem is spoiled. This is natural on the lattice, because for large densities the different definitions of topological charge density diverge and the topological charge is not conserved. In what concerns the three sectors of real eigenvectors of the Wilson action, that include respectively 0, 2 and 4, they swell so much that they start to overlap. The real eigenvalues either mix and transform into complex eigenvalues (with vanishing chirality), or they carry their chirality to the wrong sector. We remark that this is not a physical problem, providing the gauge configurations with large plaquette densities are suppressed by the gauge part of the action.

We now present our criterion, which is very simple to use. **" A γ_5 Hermitean GWR solution ${}^{gwr}D$ with constant R and constructed with the Wilson ${}^wD$ action complies maximally with the index theorem if and only if ${}^{gwr}D \to \frac{2}{R}\theta(\lambda - \lambda_0)$ when we replace ${}^wD, {}^wD^\dagger \to \lambda$, where λ is a real number, $0 \le \lambda \le 4$ and λ_0 belongs to a well determined and narrow subinterval of $]0, 2[$ ".** Our criterion also applies to GWR solutions constructed with any other action ${}^oD$ (other than the Wilson action), that

approximately complies with the index theorem and that is γ_5 hermitian, ${}^oD^\dagger = \gamma_5\,{}^oD\,\gamma_5$.

7. CONCLUSION

This talk is devoted to clarify how far the Index Theorem can be extrapolated from the continuum to finite lattices with finite topological charge densities.

The $d = 2$ Wilson action ${}^wD$ is examined, and we find that it approximately complies with the Index Theorem for finite topological charge densities lower $|\rho_1| = 0.22$. We also study the GWR solution of Neuberger which exactly complies with the Index Theorem. Finally we produce a criterion for GWR solutions, constructed with the Wilson action, that maximally comply with the Index Theorem.

With our criterion, it is simple to show that locality constitutes a sufficient condition for a Dirac action to comply with the index theorem together with the simplest topological charge q_1. It is clear that any local action is differentiable in the free limit, and therefore the condition $f(0) = 0$, $f(2) = 2/R$, $f(4) = 2/R$ can be extended to a neighborhood of these points. Using our criterion, this proves that any local action (GWR solution and with the correct free limit) complies with the index theorem at least in the case of small topological charge densities. The locality conjecture is correct in the case of actions built from the Wilson action, and it is probably correct in the general case. Moreover the locality condition can be improved. We aim at finding the action with the highest convergence radius in the neighborhood of the points 0, 2, 4, and in a sense this is comparable with finding the action with the highest locality.

Our criterion complies with the Chiu and Zenkin criterion [26] which states that the correct GWR solutions must have at least one eigenvalue at $2/R$. However we observe that it is possible to find a non-local action with eigenvalues at $2/R$, which does not have zero modes outside the free limit. Therefore, unlike the locality condition, the Chiu-Zenkin criterion is not a sufficient condition to force an action to comply with the Index Theorem, although it certainly constitutes a necessary condition.

Our criterion can be applied to the Neuberger GWR solution defined with $R\,{}^nD = I + ({}^wD - m_0)/\sqrt{({}^wD - m_0)^\dagger\,({}^wD - m_0)}$. With the λ substitution we find that ${}^nD \to \frac{2}{R}\theta(\lambda - m_0)$. This complies with our criterion when $m_0 = \lambda_0$, and it is quite evident that this is the simplest action which complies with it. The criterion shows that the Neuberger overlap action is the simplest GWR solution constructed with the Wilson action that maximally complies with the Index Theorem.

Moreover, in the Schwinger model and using the dimensionless units of $a = r = 1$, we find that $1.0 < \lambda_0 < 1.2$. This determines $m_0 = \lambda_0$ with a value which is not very far from the $m_0 \simeq 0.8$ that provides the highest locality in the free limit. This choice of m_0 increases the precision of the observation of Chandrasekharan [13], stating that the Neuberger action only complies with the Index Theorem if $0 < m_0 < 2$, ($0 < m_0 < 2$ can also be derived from the root and pole structure of the free Neuberger action). The choice of m_0 completely fixes the GWR parameter R because the free continuum limit of the lattice implies that $m_0 = a/R$. So the best choice seems to be $R \simeq 0.9a$.

Finally we outline possible continuations of this study. We are researching the analytical extension of the proof of our criterion to finite topological densities $0 << |\rho_1| < 0.22$.

The repetition of the present study in four dimensions would also be physically relevant.

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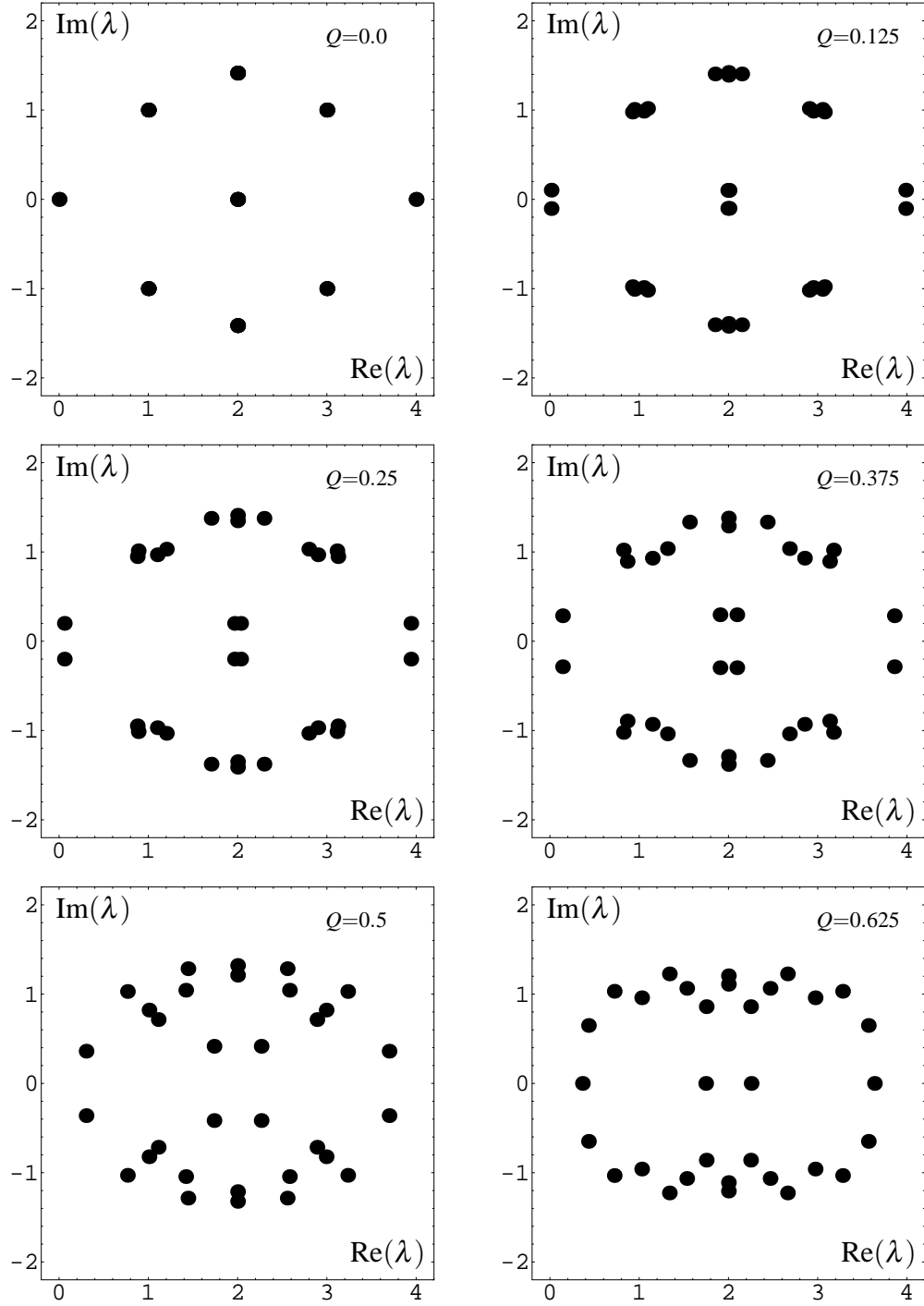26. Chiu, T., and Zenkin, S.V., Phys. Rev. **D59**, 074501 (1999) [arXiv:hep-lat/9806019].

FIGURE 1. Sequence of the eigenvalues of the Wilson action as a function of the topological parameter Q. We show the Argand plot of the eigenvalues λ of the Wilson ^{w}D, on a 4×4 lattice and with parameters $a = r = 1$.
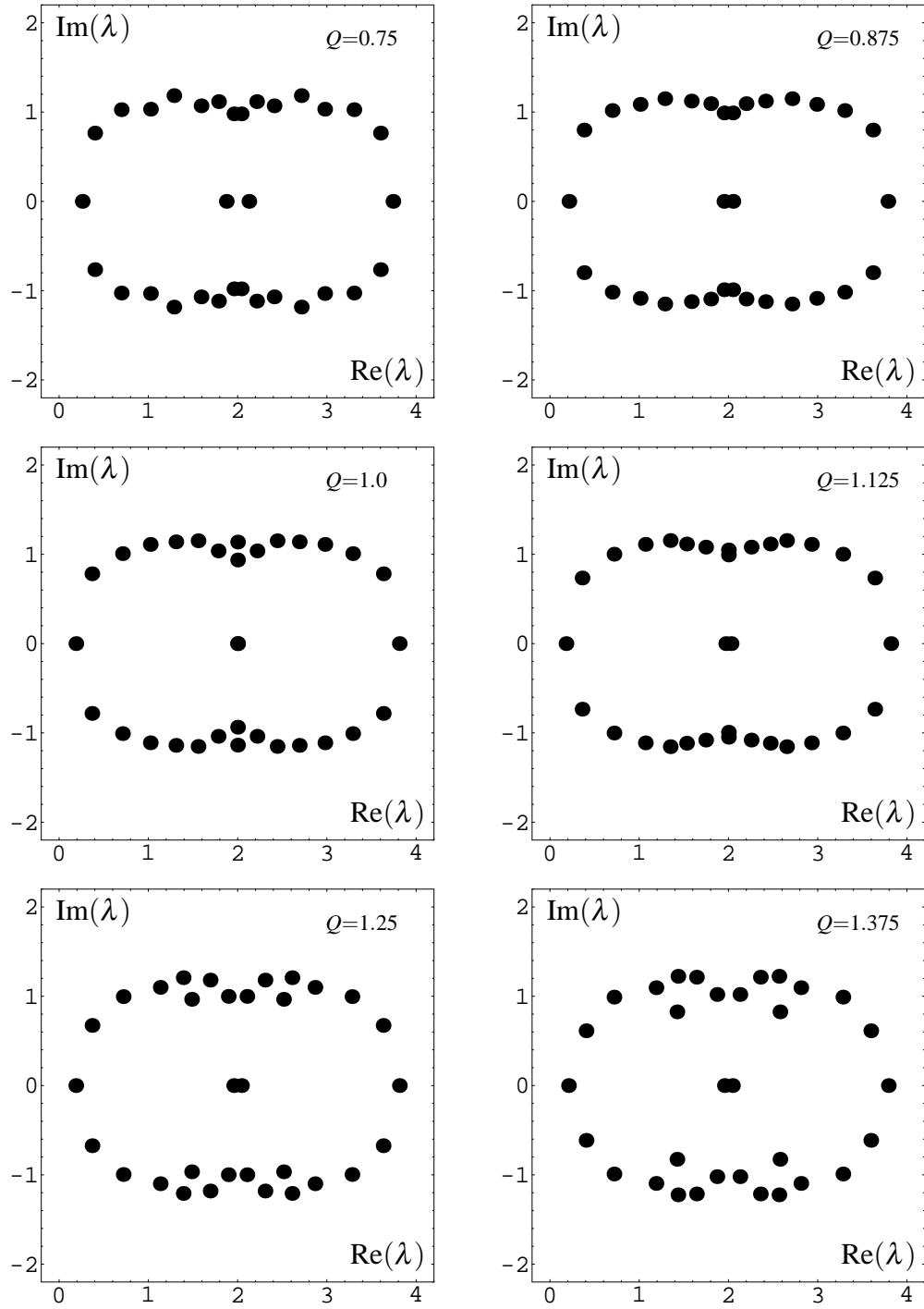
FIGURE 2. Sequence of eigenvalues of the Wilson action, continuing Fig. 2. The Wilson action successfully separates the doubling of the spectrum, not only in the free case of $Q = 0.0$, but also in topologically nontrivial cases.
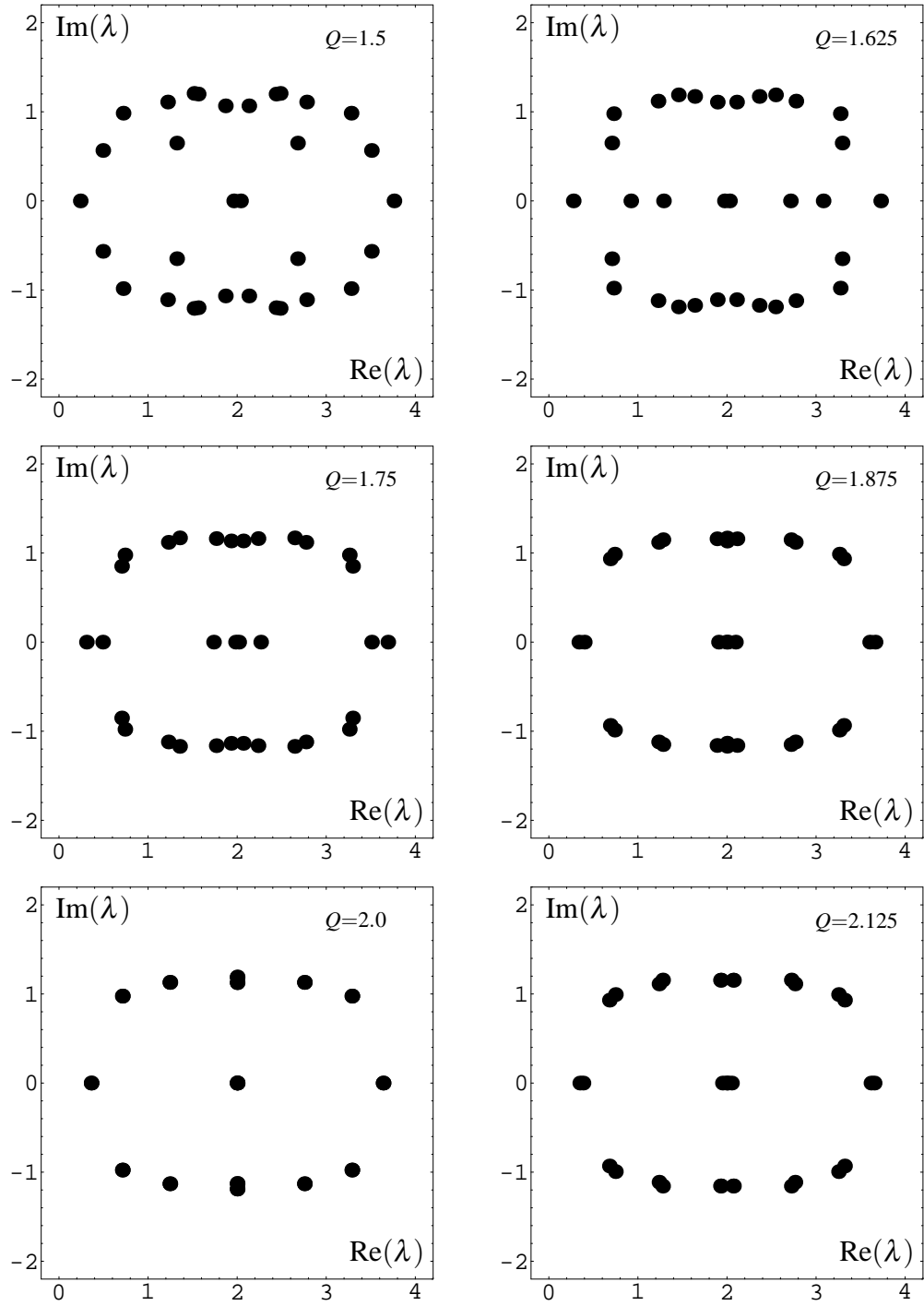
FIGURE 3. Sequence of eigenvalues of the Wilson action, continuing Figs. 1 and 2. For some particular values of Q, two opposite pairs of complex eigenvalues transform into two opposite pairs of real eigenvalues.